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金融危机与金融数学

陈增敬

山东大学

全国数学建模教学与应用会议（烟台）
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1. 前言

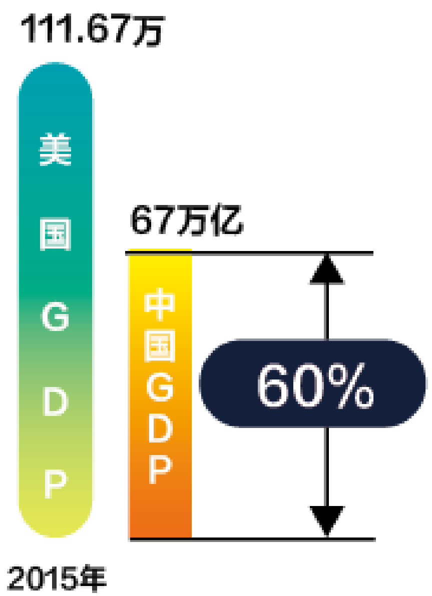
金融数学是数学与金融的交叉学科，是数学对金融的贡献也是金融对数学的贡献

危机后，金融面临着第三次金融革命，需要数学的进一步发展

2. 导言

- ⋆ 金融危机对世界经济巨大影响.防范金融风险已是关系国家安全层面的战略.各国提出了**加强金融监管**的口号.
- ⋆ **中国问题:** 中国的问题在经济,经济的问题在金融。
- ⋆ **问题:** 如何加强金融监管? 系统风险，衍生品，融资
- ⋆ **方法:**灭金融衍生产品。

债债通 Zhaizhaitong Payment Platform



中国GDP（67万亿）

是美国GDP（111.67万）的60%

货币发行量全球第一,债务总额全球第一,

资金周转率位于全球较低水平.

数据来源于国际清算银行

2.1. 金融危机的原因

金融危机的原因与金融产品创新、信用评级、会计准则、金融监管等因素和层面有着密切的关系，但是采用不当金融风险度量工具与计算方法是导致此次金融危机的重要原因之一，因此，要研究金融风险度量工具与计算方法。

⋆ **问题:**中国需要金融衍生产品？

⋆ 中国经济大国到金融强国，实现人民币的局域化和国际化，如何实施？

⋆**突破:** 金融理论,数学方法.

⋆ 现在巴塞尔协议所用的风险度量工具VaR依然可以导致新的金融危机.

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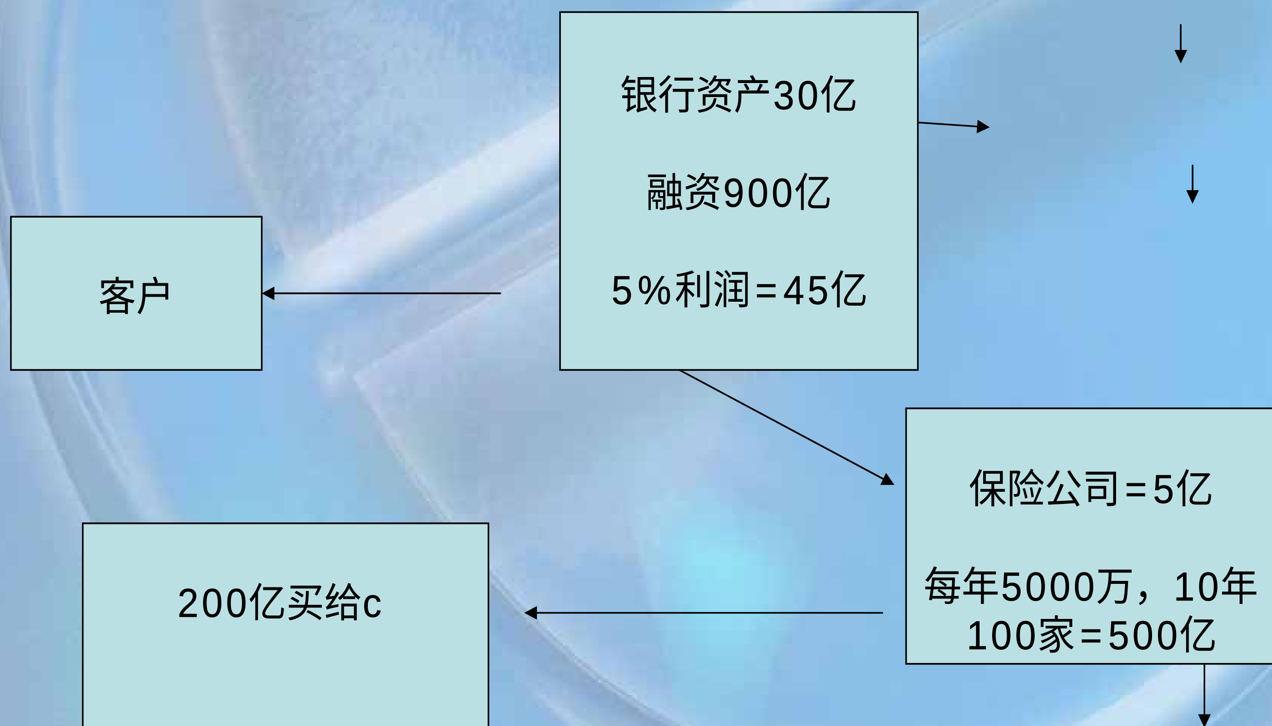

3. 现代金融的核心问题

⋆ 金融市场:债券、股票、期权、衍生品，保险、利率、汇率。

⋆ 金融问题特点: 稀缺资源进行跨时期分配，达到资源和合理配置和利用两个不同时空：现在和未来，不确定，（时间, 不确定）。

⋆ 金融技术的核心问题: 一个中心两个基本点
风险度量、资产定价、(量化)投资组合(Portfolio).

⋆ 理论：
理性预期(Markowitz-Lucas–Sargent) $\inf_x E[(X_{t+1}-x)^2|\mathcal{F}_t], x = E_P[X_{t+1}|\mathcal{F}_t]$
市场有效(Fama–Samuelson)
投资组合理论(Markowitz)（52-59）第一次金融革命。
资产定价理论(Black-Scholes)（69-73) 第二次金融革命

⋆ 风险度量(risk measure)工具: 概率统计。
概率，数学期望，独立，正态分布。
特点：可重复, 随机性. (不确定性, Uncertainty, ambiguity)

金融理论实践+技术 ⇒ 金融衍生产品⇒ 虚拟金融

⋆ 金融数学就是从不确定中找确定

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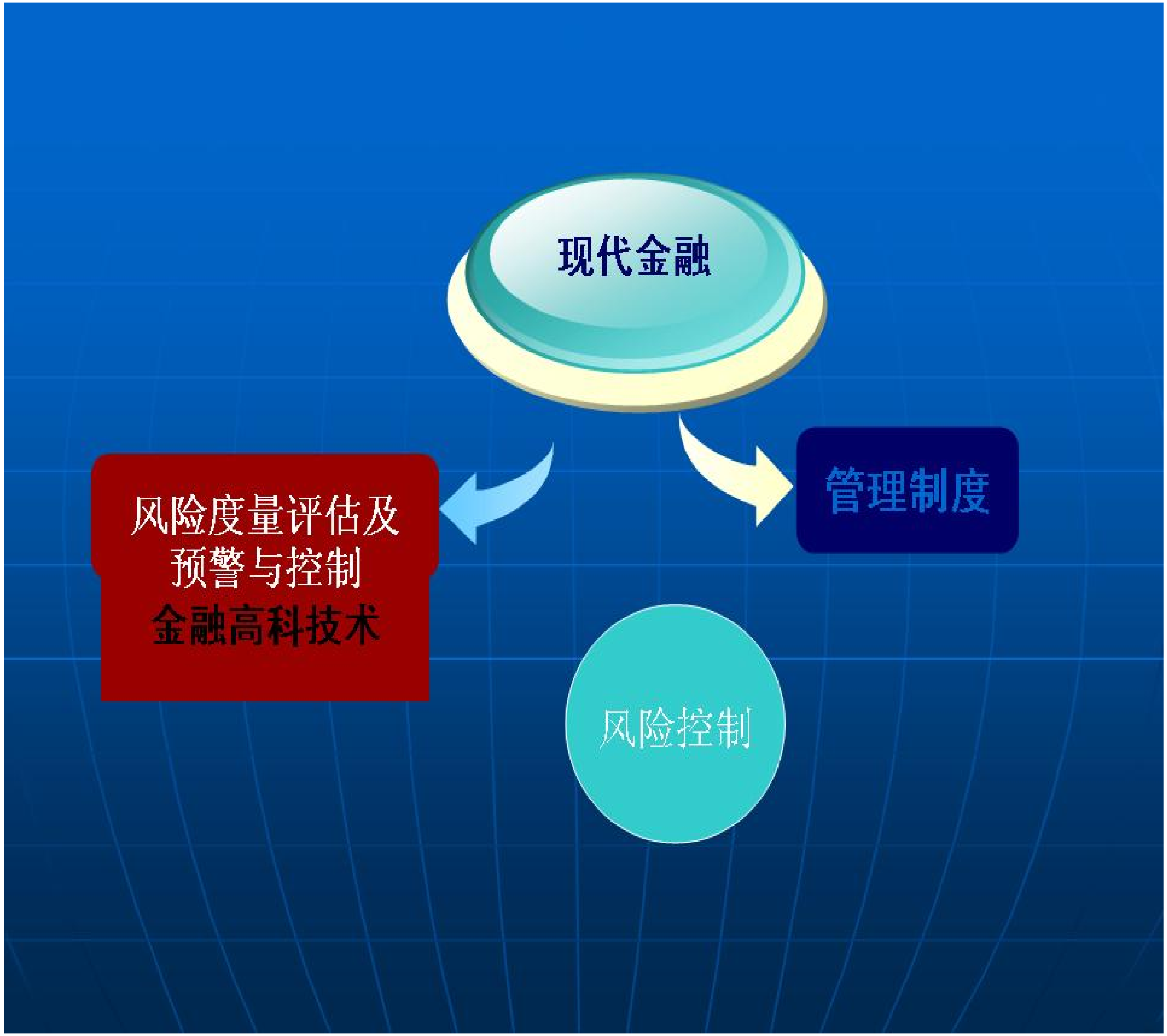


4. 资产定价理论

THEOREM 1 *(Black-Scholes, 1973:)* 在完备市场中，存在唯一的一个*neutral* 概率测度P, 使得在时刻T，未定权益ξ 的价格为$E_P[\xi e^{-rT}]$. 其中r 是债券的利率.

$\star$ 经典数学期望$\leftarrow$ Black-Scholes $\rightarrow$ 完备市场

$\star$ 计算：偏微分程，Monte Carlo。

大数定理：$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i = E_P X_1$，投资组合计算。

$\star$市场不完备, P不唯一

最大价格$max_{P\in\mathcal{P}}E_P[\xi]$

最小价格$min_{P\in\mathcal{P}}E_P[\xi]$. 非线性数学期望。

如何计算？投资组合?

$\star$ 问题： 概率论或现代数学是不是可以很好地用在不完备市场的定价问题?

观点： 现代数学已不能满足金融发展的需要。

5. 金融危机后的思考

⋆ 风险度量工具：概率统计，VAR

⋆ 市场需要新的量化投资工具和量化方法:

(1) 金融产品的猛增，财富高度集中,突发性（羊群、传染）破坏市场的独立性，市场不是正态分布，对冲基金破坏了市场均衡的原则。

(2) 理性假设的错误，不是热点清醒、冷点谨慎，而是人之弱，贪娈与恐惧：喜欢风险，厌恶风险

(3)需要新的金融理论和数学方法。概率统计不能满足需要。

概率由线性到非线性是发展的必然

保险业＋ 赌资

债券—股票

→

概率论
数学期望
数理统计

→

数学： 鞅论、 随机分析、随机学
经济：理性预期 Lucas、Arrow
金融：资产定价 Black-Scholes
物理 、生物

↓

完备市场定价

期货---期权

←

数学期望
概率论

←

数理统计

↓

衍生产品
不完备市场

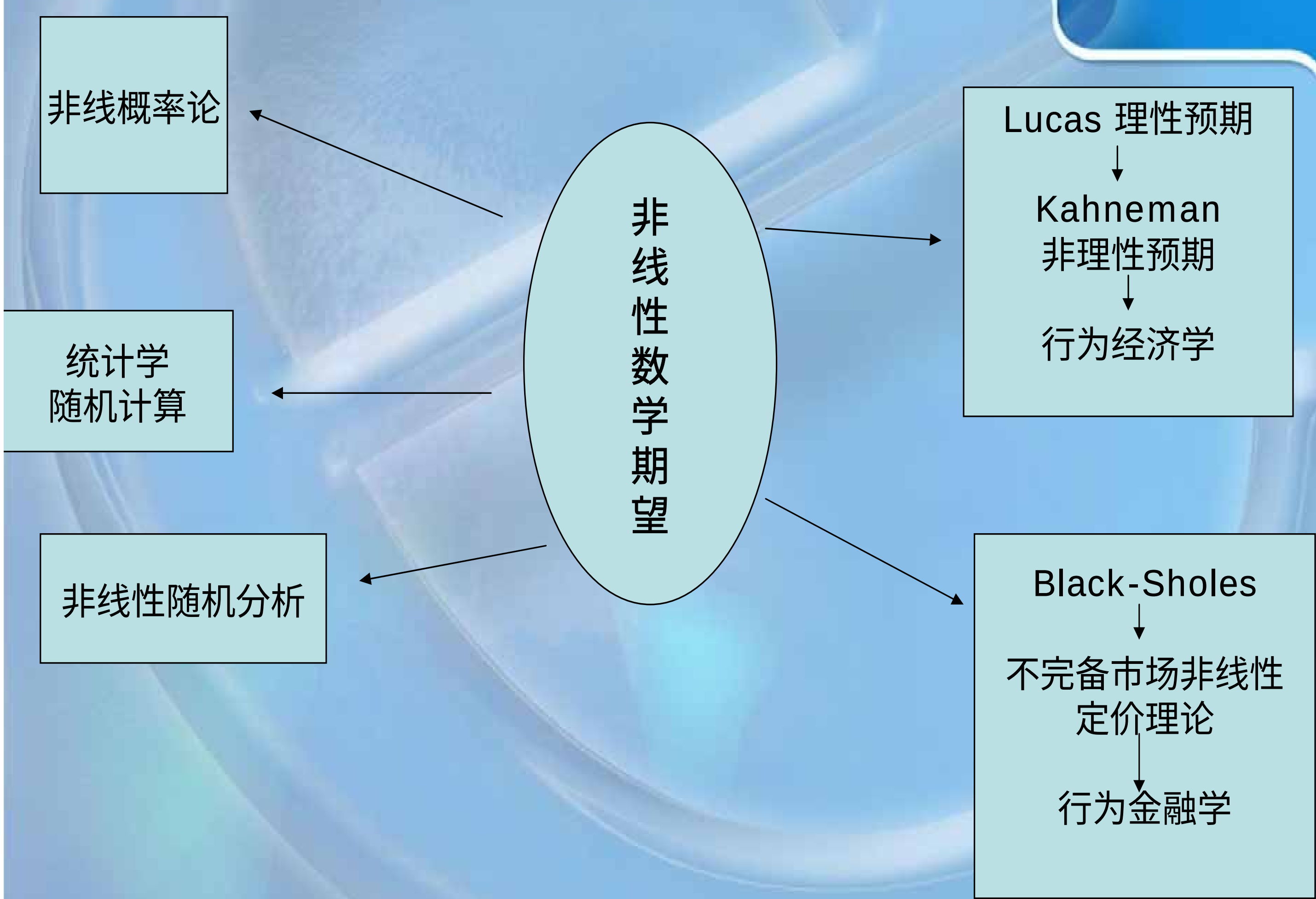
⇄

非线性期望 ？非线性概率？

→

行为经济：非理性预期
行为金融：非线资产定价

研究成果的意义


非线性数学期望
非线概率论
统计学
随机计算
非线性随机分析
Lucas 理性预期
Kahneman
非理性预期
行为经济学
Black-Sholes
不完备市场非线性
定价理论
行为金融学

5.1. Black-Scholes理论：如何发展？

- $\star$ 量化投资需要对投资者分类：喜欢风险，厌恶风险，一般大众(风险中性）。

  最大价格$\mathbb{E}[\xi] := \max_{P\in\mathcal{P}} E_P[\xi]$，最小价格$\mathcal{E}[\xi] := \min_{P\in\mathcal{P}} E_P[\xi]$.

  最大概率$V(A) := \max_{P\in\mathcal{P}} P(A)$, 最小概率$v(A) := \min_{P\in\mathcal{P}} P(A)$

- $\star$ 现在数学（概率论）不能满足金融发展的需要，必须根据金融的需要创新的数学理论和方法。

如何发展概率统计？非线性概率统计？

6. 概率论历史回顾

(1) 300年以来，概率统计对社会和科学发展的作用。Bernoulli,

(2) Bernoulli (1713): Ars congjectandi: Golden theorem, Bernoulli 模型, Aleatory(主观观概率）和Epistemic(客观概率）

(4) Keynes (1920): A treatise on probability.

(3) Kolmogorov (1933): Foundations of the theory of probability. Modern version

"From Bernoulli to Borel, some big names in the history of the LLN did not believe that probabilities should necessarily satisfy the additivity property $A \bigcap B = \emptyset \Rightarrow P(A \bigcup B) = P(A) + P(B)$. Jakob Bernoulli espoused these ideas in the same book where he presented the embryo of the LLN. In an ironic twist, Kolmogorov proved his modern version of the LLN in his book, whose enormous influence effaced the notion of non-additivity from mainstream probability,.....,mathematical finance makes a recent visibility peak of this otherwise underground trend （Pedro Teran, 2011)

7. 概率统计的美好回忆

(1) 最美的性质：线性, 可重复性

$$\text{概率期望合一} \Leftarrow \begin{cases} P(A+B) = P(A) + P(B) \Leftrightarrow E[\xi+\eta] = E[\xi] + E[\eta] \\ \text{分布函数} \Leftrightarrow \text{特征函数}, Levy \end{cases}$$

(3) 最智慧概念：独立概念

(3 最理想状态：正太分布, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

(4) 最简单模型：Bernoulli：$\Rightarrow$ 大数，中心极限和重对数律

(5) 主观客观统一：$P(A) = \lim\limits_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i$ 罐子模型：红球+黑球=100。

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \right\}_{\omega\in\Omega} \to P_R, \quad n \to \infty$$

Uncertainty disappears when there are a lager number of observations. 随着试验次数的逐步增加，不确定性渐渐消失，这就是客观概率。

7.1. 中心极限定理、大数定理、重对数率

In probability theory, the most familiar and elementary model is **Bernoulli trials**.

Assumption: $\{X_i\}$ IID , $S_n := \sum_{i=1}^n X_i$, $EX_1 = 0$, $EX_1^2 = \sigma^2$ Then

Theorem 1: Lindeberg: Mean Zero,

$$P\left(\frac{S_n}{\sqrt{n}} < x\right) \to \Phi(x/\sigma)$$

Theorem 2: Kolmogorov: $\mu := EX_1$

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu\right) = 1, \quad \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| < \epsilon\right) = 1$$

Theorem 3: Hartman–Wintner(1941): If Mean zero, $EX_1 = 0$, $EX_1^2 = \sigma^2$, Then

$$P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = \sigma\right) = 1.$$

8. 例1：不确定的多罐子模型（ambiguity）与金融

Binomial model:

$\star\, n - Urns$

$\star$ 红 + 黑 $= N_i, \quad i = 1, 2, \cdots n$

$\star$ 红 $\in [\frac{N_i}{3}, \frac{N_i}{2}]$

每个罐子取一个 $X_i = \begin{cases} 0 & \text{红} \\ 1 & \text{黑} \end{cases}$

$$\lim_{n\to\infty} \frac{S_n}{n} =?, \quad \lim_{n\to\infty} \frac{S_n}{\sqrt{n}} =?, \quad \lim_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} =?$$

相信罐子是相同（identical）或相似（indistinguishable）？独立自然成立。

identical $\Rightarrow \lim_{n\to\infty} \frac{S_n}{n} = P_R$.

Unfortunately, there is no reason to be confident that those urns are identical as if the balls were repeatedly drawn from the same urn.

9. 例2：数论中一个问题，Trinomial model

⋆ 考虑十进制首位数是1 的整数n 构成的集合：

$$A := \bigcup_{k=0}^{\infty} \{n : 10^k \leq n < 2 \cdot 10^k\}$$

$$X_i = \begin{cases} 1 & \text{如果首位是偶数}, 2、4、6、8; \\ 0 & \text{首位是}1\text{；} \\ -1 & \text{首位是奇数}, 3、5、7、9. \end{cases}$$

数论已证：$P(A) \in [\frac{1}{9}, \frac{5}{9}]$.

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{n} =?, \quad \lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{\sqrt{n}} =?, \quad \lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{\sqrt{2n \log\log n}} =?$$

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10. 最简单非线性概率与期望

(1) Capacity: (v, V), where

$$v(A) = \inf_{Q \in \mathcal{P}} Q(A), \quad V(A) = \sup_{Q \in \mathcal{P}} Q(A).$$

(2) Lower-upper expectation: $\mathcal{E}[\xi]$ and $\mathbb{E}[\xi]$

$$\mathcal{E}[\xi] = \inf_{Q \in \mathcal{P}} E_Q[\xi], \qquad \mathbb{E}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi]$$

11. Ellsberg-type Binomial models

THEOREM 2 *In Ellsberg-type model, let* $\{X_i\}_{i=1}^n$ *be defined above such that* $\overline{\mu} := \mathbb{E}[X_i]$ *and* $\underline{\mu} := \mathcal{E}[X_i]$. *Set* $S_n := \sum_{i=1}^n X_i$.

(I) Let $v(A) = \inf_{Q \in \mathcal{P}} Q(A)$, $\forall A \in \mathcal{F}$, *then for any* $\epsilon > 0$,

$$\lim_{n \to \infty} v\left(\underline{\mu} - \epsilon \leq \frac{S_n}{n} \leq \overline{\mu} + \epsilon\right) = 1.$$

(II) For any $\varphi \in C_b(\mathbb{R})$, *then*

$$\lim_{n \to \infty} \mathbb{E}\left[\varphi\left(\frac{S_n}{n}\right)\right] = \sup_{\underline{\mu} \leq x \leq \overline{\mu}} \varphi(x).$$

(III) Moreover, I $\Leftrightarrow$ *II.*

(IV) Iff $\mathcal{E}[X_1] = \mathbb{E}[X_1]$ *that*

$$v\left(\lim_{n \to \infty} \frac{S_n}{n} = \mathcal{E}[X_1]\right) = 1.$$

This theorem implies that if $\mathcal{E}[X_1] \neq \mathbb{E}[X_1]$, *then* $\lim_{n \to \infty} \frac{S_n}{n}$ *does not exists.*

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12. CLT for 数论模型/Ellsberg-type models

⋆ Upper expectation $\mathbb{E}$, $\{X_i\}$ Ellsberg-type random variables.

$\mathbb{E}[X_1] = \mathcal{E}[X_1] = 0$. Variance $\mathbb{E}[X_1^2] = \overline{\sigma}^2, \mathcal{E}[X_1^2] = \underline{\sigma}^2$.

$V(A) := \mathbb{E}[I_A]$, $v(A) = 1 - V(A^c)$.

Then

⋆ Our result:

$$V\left(\frac{S_n}{\sqrt{n}} < y\right) \to \Phi(\frac{y}{\underline{\sigma}})I_{(y>0)} + \Phi(\frac{y}{\overline{\sigma}})I_{(y\le 0)}$$

$$v\left(\frac{S_n}{\sqrt{n}} < y\right) \to \Phi\left(\frac{y}{\overline{\sigma}}\right)I_{(y>0)} + \Phi\left(\frac{y}{\underline{\sigma}}\right)I_{(y\le 0)}$$

where $S_n := \sum_{i=1}^n X_i$, Variance Uncertainty.

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13. Distribution of Ellsberg-type random variables

Let $M := \max\left\{\Phi\left(\frac{b}{\underline{\sigma}}\right) - \Phi\left(\frac{a}{\underline{\sigma}}\right), \Phi\left(\frac{b}{\overline{\sigma}}\right) - \Phi\left(\frac{a}{\overline{\sigma}}\right)\right\}.$

(I) If $b \geq a > 0$,

$$M \leq \lim_{n\to\infty} V\left(a \leq \frac{S_n}{\sqrt{n}} < b\right) \leq \Phi\left(\frac{b}{\underline{\sigma}}\right) - \Phi\left(\frac{a}{\overline{\sigma}}\right)$$

(II) If $0 \geq b \geq a$,

$$M \leq \lim_{n\to\infty} V\left(a \leq \frac{S_n}{\sqrt{n}} < b\right) \leq \Phi\left(\frac{b}{\overline{\sigma}}\right) - \Phi\left(\frac{a}{\underline{\sigma}}\right)$$

(III) If $b \geq 0 \geq a$,

$$\lim_{n\to\infty} V\left(a \leq \frac{S_n}{\sqrt{n}} < b\right) = \Phi\left(\frac{b}{\underline{\sigma}}\right) - \Phi\left(\frac{a}{\underline{\sigma}}\right)$$

14. 困难所在

The characteristic function is indispensable for the study of general limit theorems. Such a tool does not work in the nonlinear case.

$\star$ Bernolli proved for special case: Binomial distribution.

$\star$ Lindeberg: Semi-group, Stein method to prove CLT.

$\star$ Levy: Characteristic function: LLN and CLT.

$\star$ Peng:PDE

Is there any method to prove LLN without Characteristic function even in probability theory?

14.1. Remark on Bernoulli-type and Ellsberg-type

⋆ Bernuolli-type:

Let S_n be the total number of red balls drawn in n-experiments

$$\left\{ \frac{S_n(\omega)}{n} \right\}_{\omega\in\Omega} \to P_R, \quad n \to \infty$$

Uncertainty disappears when there are a lager number of observations.

⋆ Ellsberg-type:

$$\left\{ \frac{S_n(\omega)}{n} \right\}_{\omega\in\Omega} \to \left[\frac{1}{3}, \frac{1}{2}\right], \quad n \to \infty$$

Althoug Knightian uncertainty does not disappear even there are a large number of observations, but uncertainty can be decreased. That is ambiguity.

如何从古典概型到现代发展困难

(1)Choquet expectations (Choquet 1953, physics)

$$C_V[X] := \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 [V(X \geq t) - 1]dt.$$

(2)g-expectation(Peng 1997): $\mathbb{E}_g[\varphi(\xi)] := y_0$

$$y_t = \varphi(\xi) + \int_t^T g(y_s, z_s, s)ds - \int_t^T z_s dB_s$$

(3) Sub-linear expectation Delbean 等（1999），Peng（2007）等.

(a)Monotonicity: $X \geq Y$ implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

(b)Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$.

(c)Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(d)Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

$V(A) := \mathbb{E}[I_A]$ and $v(A) := 1 - V(A^c)$.

(4) 不能用特征函数，如何引进独立？

15. 现状

Keynes (1921) constructs a theory of imprecise probability. Many researchers in mathematical finance and robust statistics were attracted to the frequentist properties of random variables under non-additive (lower and upper) probabilities. Consequently, various generalizations of LLN for non-additive probabilities have been established by two groups of researchers.

(1) 非线性概率组: people tend to approach the frequentist properties under non-additive (imprecise) probabilities.

Dow and Werlang (1994), Walley and Fine(1982), Cooman and Miranda(2008), Epstein and Schneider(2003), Maccheroni and Marinacci(1999) , Chen, Wu and Li(2013), Tern(2014).

(2) 非线性期望组: people tend to investigate the frequentist properties under sub-linear(lower and upper) expectations.

Peng (2007).

16. Independence

Linear: Independence of events A and B independent $P(A/B) = P(A)$, 独立增量，金融中，今天可独立明天，但明天应该依赖今天。

Nonlinear case:

$\star$ Marinacci's preindependent (2005): $V(AB) = V(A)V(B)$.

$\star$ Puhalskii independent(2001): $v(AB) = v(A)v(B)$.

$\star$ Epstein orthogonal independence: $\mathbb{E}[g(\xi)f(\eta) = \mathbb{E}[g(\xi)\mathbb{E}[f(\eta)]$. f and g 非负。
以上定义主要适合Choquet-期望.

Peng's independence：A random variable $X \in \mathcal{H}$ is said to be independent of Y under $\mathbb{E}$, if for each φ such that $\varphi(X,Y) \in \mathcal{H}$ and $\varphi(x,Y) \in \mathcal{H}$ for each $y \in \mathbb{R}$

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\overline{\varphi}(Y)],$$

where $\overline{\varphi}(x) := \mathbb{E}[\varphi(x,Y)]$. Marginal expectation.

17. LLN and CLT for sub-linear expectations

$\star$ Maximum distribution :

THEOREM 3 *(Peng 2007,2008)* $\{X_i\}_{i=1}^{\infty}$ *IID random variables,* $\overline{\mu} := \mathbb{E}[X_1]$, $\underline{\mu} := \mathcal{E}[X_1]$. *Then for any continuous and linear growth function* ϕ,

$$\mathbb{E}\left[\phi\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] \to \sup_{\underline{\mu}\le x\le\overline{\mu}} \phi(x), \quad \textit{as } n\to\infty.$$

$\star$ CLT :

THEOREM 4 *(Peng 2006, 2008)* $\{X_n\}$ *IID, zero means* $\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$, *finite variance* $\mathbb{E}[X_1^2] = \overline{\sigma}^2$, $-\mathbb{E}[-X_1^2] = \underline{\sigma}^2$, *Then,*

$$\mathbb{E}[\phi(S_n/\sqrt{n})] \to \mathbb{E}[\phi(\xi)]$$

where ξ *is* G*-normal under* $\mathbb{E}[\cdot]$. *Method: PDE.*

18. Convolution and independence

Both convolution and independence are important concepts in the development of probability theory.

Let ξ and η be two random variables on probability space $(\Omega, \mathcal{F}, P)$, and $E_P[\cdot]$ be the expectation operator under probability P. Also let $u(x)$ and $E_P[\varphi(x+\eta)]|_{x=\xi}$ given by

$$u(x) := E_P[\varphi(x+\eta)], \quad E_P[\varphi(x+\eta)]\big|_{x=\xi} := u(\xi).$$

ξ and η are said to be of convolution if for any function φ,

$$E_P\left[\varphi(\xi+\eta)\right] = E_P\left[E_P\left[\varphi(x+\eta)\right]\big|_{x=\xi}\right].$$

Because of the linearity of linear expectation, Fubini theorem implies that the order of integrations is exchangeable.Thus

$$\begin{aligned} E_P[\varphi(\xi+\eta)] &= E_P\left[E_P[\varphi(x+\eta)]\big|_{x=\xi}\right] \\ &= E_P\left[E_P[\varphi(\xi+y)]\big|_{y=\eta}\right]. \end{aligned} \tag{1}$$

18.1. Convolution and Independence

Suppose that $C_b(\mathbb{R})$ ($C_b^+(\mathbb{R})$) is the set of all (positive) continuous and bounded functions on $\mathbb{R}$ and $C_b^2(\mathbb{R})$ is the set of all continuous and bounded functions on $\mathbb{R}$ whose second derivatives exist in $C_b(\mathbb{R})$.

DEFINITION 1 *Let X and Y be two random variables in $\mathcal{H}$.*

1. φ-**Convolution:** *Given a function $\varphi \in C_b(\mathbb{R})$, random variable Y is said to be φ-convolution of X under sub-linear expectation $\mathbb{E}$, if for all constants $z, a, b \in \mathbb{R}$,*

$$\mathbb{E}[\varphi(z+aX+bY)] = \mathbb{E}\left[\mathbb{E}[\varphi(x+bY)]\big|_{x=z+aX}\right].$$

2. **Common first moment:** *Random variables X and Y are said to be of finite common first moment, if $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$ such that*

$$\mathcal{E}[\varphi(X)] = \mathcal{E}[\varphi(Y)], \quad \mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)], \text{ when } \varphi(x) = x \text{ or } |x|.$$

3. φ-**Convolutionary sequence:** *Given $\varphi \in C_b(\mathbb{R})$, a sequence $\{X_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ is said to be of φ-convolution, if X_{i+1} is φ-convolutionary of $\sum_{j=1}^{i} X_j$ for any $i = 1, 2, \cdots$.*

19. Main results-1

THEOREM 5 **LLNs:** *Let $\mathbb{E}$ be a sub-linear expectation and $\mathcal{E}$ be its conjugate expectation. Assume that $\{X_i\}_{i=1}^{\infty}$ is a sequence of random variables with finite common first moment $\mathbb{E}[X_i] = \overline{\mu}$ and $\mathcal{E}[X_i] = \underline{\mu}$. Set $S_n := \frac{1}{n}\sum_{i=1}^{n} X_i$. If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[|X_i| I_{\{|X_i|>n\epsilon\}}\right] = 0, \tag{2}$$

then

(I) For any positively bounded function $\varphi \in C_b^2(\mathbb{R})$, if $\{X_i\}_{i=1}^{\infty}$ is a sequence of φ-convolutionary random variables under $\mathbb{E}$, then

$$\lim_{n\to\infty} \mathbb{E}\left[\varphi\left(\frac{S_n}{n}\right)\right] = \sup_{\underline{\mu}\le x\le\overline{\mu}} \varphi(x), \forall \varphi \in C_b(\mathbb{R}) \tag{3}$$

(II) If for any positively monotone function $\varphi \in C_b^2(\mathbb{R})$, $\{X_i\}_{i=1}^{\infty}$ is a sequence of φ-convolutionary random variables under $\mathbb{E}$, then for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} v\left(\underline{\mu} - \epsilon \le \frac{S_n}{n} \le \overline{\mu} + \epsilon\right) = 1.$$

20. Main results-2

A sufficient condition under which the maximal distribution is equivalent to a weak law of large number for capacity.

THEOREM 6 *Suppose that* $\mathbb{E}$ *is a sub-linear expectation and* $\mathcal{E}$ *is its conjugate expectation. Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of* φ*-convolution random variables for all* $\varphi \in C_b^+(\mathbb{R})$ *with finite common first moment* $\mathbb{E}[X_i] = \overline{\mu}$ *and* $\mathcal{E}[X_i] = \underline{\mu}$ *such that condition (12) holds. Set* $S_n := \sum_{i=1}^n X_i$*. Then, the followings are equivalent.*

(I) For any $\epsilon > 0$,

$$\lim_{n\to\infty} v\left(\underline{\mu} - \epsilon \le \frac{S_n}{n} \le \overline{\mu} + \epsilon\right) = 1.$$

(II) For any $\varphi \in C_b(\mathbb{R})$,

$$\lim_{n\to\infty} \mathbb{E}\left[\varphi\left(\frac{S_n}{n}\right)\right] = \sup_{\underline{\mu} \le x \le \overline{\mu}} \varphi(x).$$

21. Main results-3

THEOREM 7 $\{X_i\}_{i=1}^n$ *is of* φ*-convolution under nonlinear expectation* $\mathbb{E}$ *for all* $\varphi \in C_b^+(R)$. *Set* $\overline{\mu} := \mathbb{E}[X_i]$, $\underline{\mu} := \mathcal{E}[X_i]$ *and* $S_n := \sum_{i=1}^n X_i$. *If* $\mathbb{E}[|X_i|^{1+\alpha}] < \infty$ *for* $\alpha > 0$. *Then*

(I)

$$v\left(\underline{\mu} \leq \liminf_{n\to\infty} S_n/n \leq \limsup_{n\to\infty} S_n/n \leq \overline{\mu}\}\right) = 1.$$

(II)

$$V\left(\limsup_{n\to\infty} S_n/n = \overline{\mu}\right) = 1$$

$$V\left(\liminf_{n\to\infty} S_n/n = \underline{\mu}\right) = 1.$$

22. Main results-4: SLLN without sub-linearity and independence

THEOREM 8 *Assume that* $\{X_i\}$ *has the upper-lower Choquet integrals, i.e.* $\mathbb{E}[X_i] = \overline{\mu}$ *and* $\mathcal{E}[X_i] = \underline{\mu}$ *for each* $i \geq 1$. *Set* $S_n := \sum_{i=1}^n X_i$.

(i) If

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[\exp\left(\frac{a}{\sqrt{n}}\sum_{i=1}^n X_i\right)\right]}{\prod_{i=1}^n \mathbb{E}\left[\exp\left(\frac{a}{\sqrt{n}}X_i\right)\right]} < \infty, \qquad a = \pm 1. \tag{4}$$

Then

$$v\Big(\underline{\mu} \leq \liminf_{n\to\infty} \frac{S_n}{n} \leq \limsup_{n\to\infty} \frac{S_n}{n} \leq \overline{\mu}\Big) = 1. \tag{5}$$

(ii) Iff

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[\exp\left(\frac{a}{\sqrt{n}}\sum_{i=1}^n X_i\right)\right]}{\prod_{i=1}^n \mathbb{E}\left[\exp\left(\frac{a}{\sqrt{n}}X_i\right)\right]} \geq 1, \qquad a = \pm 1. \tag{6}$$

Then, the interval $[\underline{\mu}, \ \overline{\mu}]$ *is the unique smallest interval such that equality (2) holds.*

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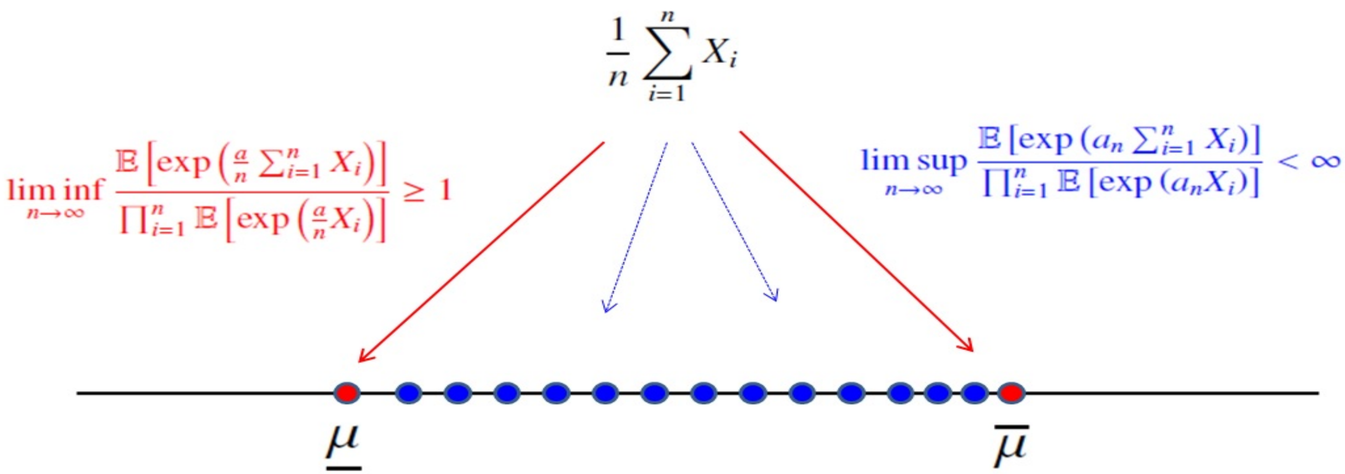

Figure 1: The interpretation of the sample mean

- Subjective interpretation

$$V\Big(\limsup_{n\to\infty}\frac{S_n}{n}=\overline{\mu}\Big)=1$$

- Objective interpretation

$$v\Big(\underline{\mu}\leq\liminf_{n\to\infty}\frac{S_n}{n}\leq\limsup_{n\to\infty}\frac{S_n}{n}\leq\overline{\mu}\Big)=1.$$

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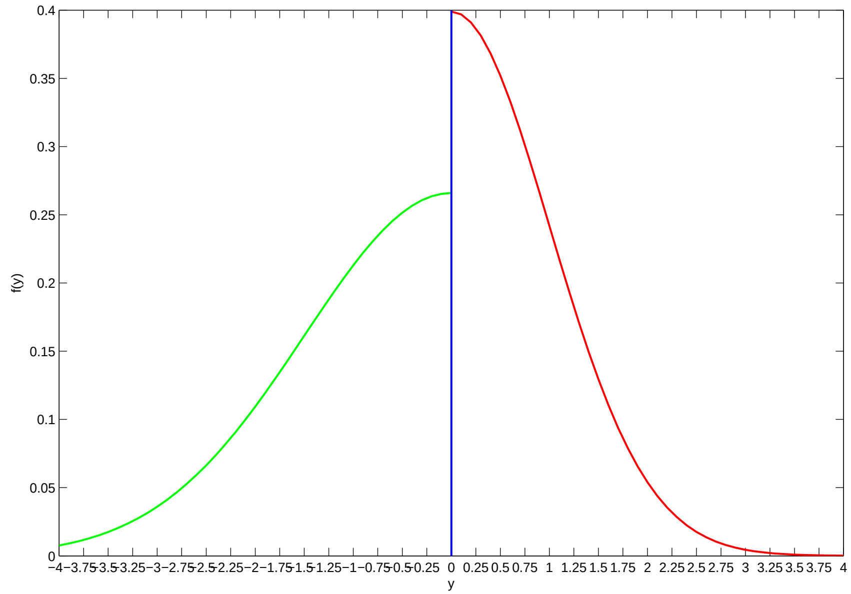
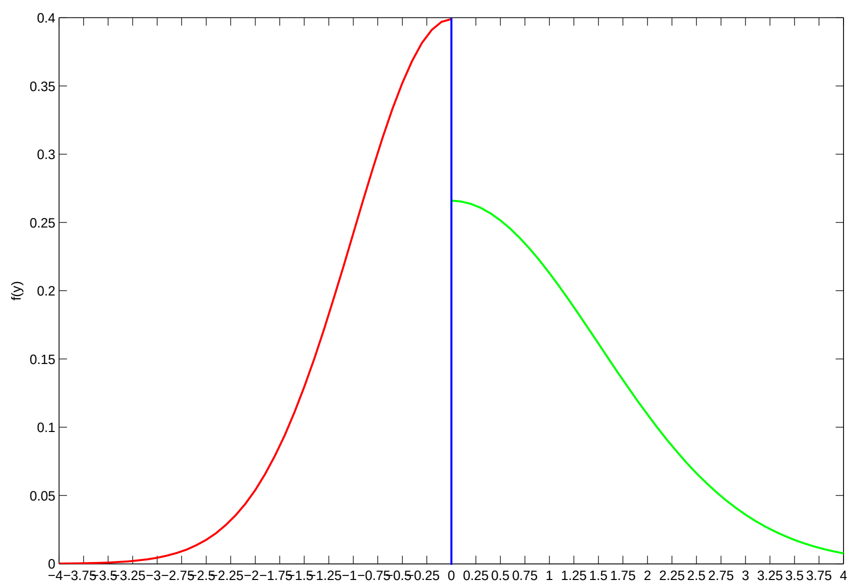


Figure 1: limited -p.d.f. of — with — , — , —



23. Main Lemma

LEMMA 1 *Suppose* φ *is a bounded function on* $\mathbb{R}$. *Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of* φ*-convolutionary random variables under a sub-linear expectation* $\mathbb{E}$. *Then for any constant* $y_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$,

$$I_1(n) \leq \mathbb{E}\left[\varphi\left(\sum_{i=1}^{n} X_i\right)\right] - \varphi\left(\sum_{i=1}^{n} y_i\right) \leq I_2(n), \tag{7}$$

where

$$I_1(n) := \sum_{m=1}^{n} \inf_{x\in\mathbb{R}} \left\{\mathbb{E}\left[\varphi\left(x + X_{n-(m-1)} - y_m\right)\right] - \varphi\left(x\right)\right\},$$

$$I_2(n) := \sum_{m=1}^{n} \sup_{x\in\mathbb{R}} \left\{\mathbb{E}\left[\varphi\left(x + X_{n-(m-1)} - y_m\right)\right] - \varphi\left(x\right)\right\}.$$

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24. Proof of Main results-1

Proof Set $S_n := \sum_{i=1}^n X_i$ with $S_0 = 0$.

$$
\begin{aligned}
&\mathbb{E}[\varphi(S_n)] - \varphi\left(\sum_{i=1}^n y_i\right) \\
&= \mathbb{E}[\varphi(S_n)] - \mathbb{E}[\varphi(S_{n-1} + y_1)] \\
&\quad + \mathbb{E}[\varphi(S_{n-1} + y_1)] - \mathbb{E}\left[\varphi\left(S_{n-2} + \sum_{i=1}^2 y_i\right)\right] + \cdots \\
&\quad + \mathbb{E}\left[\varphi\left(S_{n-m} + \sum_{i=1}^m y_i\right)\right] - \mathbb{E}\left[\varphi\left(S_{n-(m+1)} + \sum_{i=1}^{m+1} y_i\right)\right] + \cdots \\
&\quad + \mathbb{E}\left[\varphi\left(S_1 + \sum_{i=1}^{n-1} y_i\right)\right] - \mathbb{E}\left[\varphi\left(\sum_{i=1}^n y_i\right)\right] \\
&= \sum_{m=0}^{n-1}\left\{\mathbb{E}\left[\varphi\left(S_{n-m} + \sum_{i=1}^m y_i\right)\right] - \mathbb{E}\left[\varphi\left(S_{n-(m+1)} + \sum_{i=1}^{m+1} y_i\right)\right]\right\}. \qquad (8)
\end{aligned}
$$

We now evaluate each term inside the summation.

Let $h(x) := \mathbb{E}\left[\varphi(x + X_{n-m})\right]$. Then by the convolution of $\{X_i\}_{i=1}^n$,

$$\begin{aligned}\mathbb{E}\left[\varphi\left(S_{n-m} + \sum_{i=1}^m y_i\right)\right] &= \mathbb{E}\left[\mathbb{E}[\varphi(x + X_{n-m})]\big|_{x=S_{n-(m+1)}+\sum_{i=1}^m y_i}\right] \\ &= \mathbb{E}\left[h\left(S_{n-(m+1)} + \sum_{i=1}^m y_i\right)\right].\end{aligned}$$

Then by the sub-linearity of $\mathbb{E}$, we have

$$\begin{aligned}&\mathbb{E}\left[\varphi\left(S_{n-m} + \sum_{i=1}^m y_i\right)\right] - \mathbb{E}\left[\varphi\left(S_{n-(m+1)} + \sum_{i=1}^{m+1} y_i\right)\right] \\ = \ &\mathbb{E}\left[h\left(S_{n-(m+1)} + \sum_{i=1}^m y_i\right)\right] - \mathbb{E}\left[\varphi\left(S_{n-(m+1)} + \sum_{i=1}^m y_i + y_{m+1}\right)\right] \\ \leq \ &\mathbb{E}\left[h\left(S_{n-(m+1)} + \sum_{i=1}^m y_i\right) - \varphi\left(S_{n-(m+1)} + \sum_{i=1}^m y_i + y_{m+1}\right)\right] \\ \leq \ &\sup_{x\in\mathbb{R}}\left(h(x) - \varphi\left(x + y_{m+1}\right)\right) \\ = \ &\sup_{x\in\mathbb{R}}\left\{\mathbb{E}\left[\varphi\left(x + X_{n-m}\right)\right] - \varphi\left(x + y_{m+1}\right)\right\}.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\mathbb{E}\left[\varphi\left(S_{n-m}+\sum_{i=1}^{m} y_i\right)\right]-\mathbb{E}\left[\varphi\left(S_{n-(m+1)}+\sum_{i=1}^{m+1} y_i\right)\right]\\
=\ &\mathbb{E}\left[h\left(S_{n-(m+1)}+\sum_{i=1}^{m} y_i\right)\right]-\mathbb{E}\left[\varphi\left(S_{n-(m+1)}+\sum_{i=1}^{m} y_i+y_{m+1}\right)\right]\\
\geq\ &\mathcal{E}\left[h\left(S_{n-(m+1)}+\sum_{i=1}^{m} y_i\right)-\varphi\left(S_{n-(m+1)}+\sum_{i=1}^{m} y_i+y_{m+1}\right)\right]\\
\geq\ &\inf_{x\in\mathbb{R}}\left(h(x)-\varphi\left(x+y_{m+1}\right)\right)\\
=\ &\inf_{x\in\mathbb{R}}\left\{\mathbb{E}\left[\varphi\left(x+X_{n-m}\right)\right]-\varphi\left(x+y_{m+1}\right)\right\}.
\end{aligned}$$

Combining the above two inequalities, we have

$$\inf_{x\in\mathbb{R}}\left\{\mathbb{E}\left[\varphi\left(x+X_{n-m}\right)\right]-\varphi\left(x+y_{m+1}\right)\right\} \tag{9}$$

$$\leq \mathbb{E}\left[\varphi\left(S_{n-m}+\sum_{i=1}^{m} y_i\right)\right]-\mathbb{E}\left[\varphi\left(S_{n-(m+1)}+\sum_{i=1}^{m+1} y_i\right)\right] \tag{10}$$

$$\leq \sup_{x\in\mathbb{R}}\left\{\mathbb{E}\left[\varphi\left(x+X_{n-m}\right)\right]-\varphi\left(x+y_{m+1}\right)\right\}. \tag{11}$$

This with Eq.(8) implies the conclusion.

25. Main Lemma

LEMMA 2 *Let* $\mathbb{E}$ *be a sub-linear expectation and* $\mathcal{E}$ *be its conjugate expectation. Suppose that* $\{X_i\}_{i=1}^{\infty}$ *is a sequence of random variables with finite common first moment* $\mathbb{E}[X_i] = \overline{\mu}$ *and* $\mathcal{E}[X_i] = \underline{\mu}$ *for each* $i \geq 1$*. Moreover, for any* $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[|X_i| I_{\{|X_i|>n\epsilon\}}\right] = 0. \tag{12}$$

Then for any monotone *function* $\varphi \in C_b^2(\mathbb{R})$, *we have*

(I)

$$\lim_{n\to\infty} \inf_{\underline{\mu}\leq y_i\leq\overline{\mu}} \sum_{i=1}^{n} \sup_{x\in\mathbb{R}} \left\{\mathbb{E}\left[\varphi\left(x+\frac{X_i-y_i}{n}\right)\right] - \varphi(x)\right\} = 0;$$

(II)

$$\lim_{n\to\infty} \inf_{\underline{\mu}\leq y_i\leq\overline{\mu}} \sum_{i=1}^{n} \inf_{x\in\mathbb{R}} \left\{\mathbb{E}\left[\varphi\left(x+\frac{X_i-y_i}{n}\right)\right] - \varphi(x)\right\} = 0.$$

26. Law of iterated logarithm for sub-linear expectations

THEOREM 9 *$\{X_n\}$ bounded IID.* $\mathbb{E}[X_1] = \mathcal{E}[X_1] = 0$, $\overline{\sigma}^2 := \mathbb{E}[X_1^2]$, $\underline{\sigma}^2 := \mathcal{E}[X_1^2]$. *Let* $S_n := \sum_{i=1}^n X_i$, $a_n := \sqrt{2n \lg \lg n}$, *then*

(I)

$$v\left(\underline{\sigma} \leq \limsup_n \frac{S_n}{a_n} \leq \overline{\sigma}\right) = 1;$$

(II)

$$v\left(-\overline{\sigma} \leq \liminf_n \frac{S_n}{a_n} \leq -\underline{\sigma}\right) = 1.$$

(III) Suppose that $C(\{x_n\})$ *is the cluster set of a sequence of* $\{x_n\}$ *in* R, *then*

$$v\left(C(\{S_n/\sqrt{2n \mathrm{loglogn}}\}) \supset (-\underline{\sigma}, \underline{\sigma})\right) = 1.$$

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27. Remarks

Probability theory was motivated by issues related to games of chance (Pascal, Fermat, Huygens, DeMoivre, Laplace). Main Driving Force: The world is deterministic, but because of ignorance, probability theory is necessary(Alain BENSOUSSAN, 2014).
”By ‘uncertain’ knowledge, let me explain, I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty...The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years hence...About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know.” – Keynes (1937, QJE, pp. 213-214)。

$(\Omega, \mathcal{F}, P)$, P 是已知的，然而，现实中P 本身是不确定的。

28. Application

Chung(1948)(Maximum partial sums), Strassen(1964)(Invariance principle): $\{X_i\}$ IID , $S_n := \sum_{i=1}^n X_i$, $EX_1 = 0$, $EX_1^2 = 1$, then

$$P\left(\limsup_{n\to\infty} \frac{1}{\sqrt{(2n^3 \log\log n)}} \Sigma_{i=1}^n |S_i| = \frac{1}{\sqrt{3}}\right) = 1.$$

$$P\left(\limsup_{n\to\infty} \frac{1}{n^2(2\log\log n)} \Sigma_{i=1}^n |S_i|^2 = \frac{4}{\pi^2}\right) = 1.$$

Theorem: $\{X_i\}$ IID, $\mathbb{E}X_1 = \overline{\mu}$, $\mathcal{E}[X_1] = \underline{\mu}$, $\mathbb{E}|X_1|^{1+\alpha} < \infty$, then

$$V\left(\limsup_{n\to\infty} \frac{1}{n^2} \Sigma_{i=1}^n |S_i - i\frac{\overline{\mu}+\underline{\mu}}{2}| = \frac{\overline{\mu}-\underline{\mu}}{2\sqrt{3}}\right) = 1.$$

$$V\left(\limsup_{n\to\infty} \frac{1}{n^3} \Sigma_{i=1}^n |S_i - i\frac{\overline{\mu}+\underline{\mu}}{2}|^2 = \frac{(\overline{\mu}-\underline{\mu})^2}{\pi^2}\right) = 1.$$

29. Application2: Urn Models

n urns, urn i contains total N_i balls with Black,Red, Yellow (Black + Red + Yellow $= N_i$)

Suppose Black = Red, Yellow $\in [\frac{3N_i}{10}, \frac{4N_i}{10}]$, then $P_Y \in [3/10, 4/10]$.

a ball is drawn randomly from each urn,

$X_i = 1$, if ball is black,

$X_i = 0$, if ball is Yellow,

$X_i = -1$ for red.

$S_n = \sum_{i=1}^{n} X_i$, is the excess frequency of black than Red

Then

(a) $\mathbb{E}[X_i] = \mathcal{E}[X_i] = 0$

(b)

$$\sqrt{6/10} \leq \lim_{n\to\infty} \sup \frac{S_n}{\sqrt{2n \lg \lg n}} \leq \sqrt{8/10}.$$

$$-\sqrt{8/10} \leq \lim_{n\to\infty} \inf \frac{S_n}{\sqrt{2n \lg \lg n}} \leq -\sqrt{6/10}.$$

30. Application in finance

In incomplete markets, there exists a set $\mathcal{P}$ of probability measures, such that the super-sub-hedging price of option ξ at strike date T are given by $\underline{\mu} := \inf_{Q\in\mathcal{P}} E_Q[\xi], \overline{\mu} := \sup_{Q\in\mathcal{P}} E_Q[\xi]$.

then

(1)

$$\underline{\mu} \leq \liminf_{n\to\infty} S_n(\omega)/n \leq \limsup_{n\to\infty} S_n/n(\omega) \leq \overline{\mu}$$

(2)

$$\limsup_{n\to\infty} S_n(\omega)/n = \overline{\mu}, \quad V,$$

$$\liminf_{n\to\infty} S_n(\omega)/n = \underline{\mu}, \quad V$$

31. Application in The Mertens conjecture

Let $\mu(n)$ denote the M$\ddot{o}$bius function, such that

$\mu(n) = 1,$ if n is a square-free integer with an even unmber of primce factors

$\mu(n) = 0,$ if n is not square-free

$\mu(n) = -1,$ if n is a square-free integer with an odd unmber of primce factors and let

$$M(x) = \sum_{n \leq x} \mu(n)$$

Merten conjectured (1890's),

$$M(x) \leq \sqrt{x}, \qquad \text{for all sufficently large} x.$$

1985, Odlyzko gave the disproof of the conjecture in numerical method and show that

$$\limsup_{x \to \infty} \frac{M(x)}{\sqrt{x}} \geq 1.06$$

Odlyzko also proposed another conjecture

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} = \infty.$$

But the conjecture remains unproved. (the conjecture is "very probable".)

Good and Churchouse(1968) conjectured

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x \lg \lg x}} = \frac{\sqrt{12}}{\pi}.$$

under the assumption that $\mu(n)$ IID with $P(\mu(n) = 1) = P(\mu(n) = -1) = \frac{3}{\pi^2}$.

It seems very probable that Good and Churchouse is false. 4cm

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